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ON A SOLVABILITY OF LINEAR MULTIPOINT BOUNDARY VALUE PROBLEM FOR THE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

***Zh.M. Kadirbayeva*[1]**
[1]Cand. Sci (Physics,Math), Senior Lecturer,
Kazakh State Women's Teacher Training University,
Almaty, Kazakhstan, e-mail: apelman86pm@mail.ru

A linear multipoint boundary value problem for the system of ordinary differential equations is considered. The method of parameterization is used for solving the considering problem. The linear multipoint boundary value problem for the system of ordinary differential equations by introducing additional parameters at the ends and at points of partition of the interval is reduced to an equivalent boundary value problem with parameters. The article is illustrated by an example for finding the solution of the linear three-point boundary value problem for the system of ordinary differential equations.

Key words: differential equation, boundary value problem, parameterization method, solvability, fundamental matrix.

The theory of boundary value problems for ordinary differential equations is one of the actual and actively developing areas of the qualitative theory of differential equations and applied mathematics [1, 2].

One of constructive and effective methods for solving of the two-point boundary value problems for ordinary differential equations is a parameterization method [3, 4]. This method, except the proof of the unique solvability of the investigated problem, gives algorithm of constructing approximate solutions converging to its exact solution. Parametrization method has allowed establishing the necessary and sufficient conditions for the unique solvability of the problem in the terms of initial data.

The parametrization method is developed for the multipoint boundary value problems for the systems of ordinarydifferential equations [5-7], and in this works the effective solvability conditions are established, and the constructive algorithms for finding a solution are constructed.

The method of parameterization is used for solving a linear multipoint boundary value problem for the system of ordinary differential equationsin work [8]. The linear multipoint boundary value problem for the system of ordinary differential equations by introducing additional parameters at the partitioning points of interval is reduced to an equivalent boundary value problem with parameters. The equivalent boundary value problem with parameters consist of the Cauchy problem for the system of ordinary differential equations with parameters, boundary condition and condition of continuity. The solution of the Cauchy problem for the system of ordinary differential equations with parameters is constructed using the fundamental matrix of the differential equation. The system of a linear algebraic equations with respect to the parameters are composed by substituting the values of the corresponding points in the built solutions to the boundary condition and the condition of continuity. A criterion of the unique solvability of the linear multipoint boundary value problem for the system of ordinary differential equations in terms of the matrix of system of the linear algebraic equations with respect to the parameters.

In this paper, parameterization method is also used to solve the linear multipoint boundary-value problem for a system of ordinary differential equations, only here an additional parameter is also introduced at the end point of the interval. This means that the number of unknown parameters is greater by one than the number of unknown functions. This is the novelty of the proposed article.

In the present paper we consider a linear multipoint boundary-value problem for a system of ordinary differential equations:

$$\frac{dx}{dt} = A(t)x + f(t), \; t \in [0,T], \quad x \in R^n, \, (1)$$

$$\sum_{j=0}^{N} B_j x(t_j) = d, \quad d \in R^n, \; (2)$$

where $(n\times n)$-matrix $A(t)$ and n - vector-function $f(t)$ are continuous on $[0,T]$, B_j are constant $(n\times n)$ matrices, $j=\overline{0,N}$, $d-$ n-constant vector, $0=t_0<t_1<t_2<...<t_{N-1}<t_N=T$, $\|x\|=\max\limits_{i=\overline{1,n}}|x_i|$, $\|A(t)\|=\max\limits_{i=\overline{1,n}}\sum\limits_{j=1}^{n}|a_{ij}(t)|\le\alpha,\ \alpha-const$.

$C([0,T],R^n)$ is the space of continuous functions $x:[0,T]\to R^n$ with the norm $\|x\|_1=\max\limits_{t\in[0,T]}\|x(t)\|$.

A solution of problem (1), (2) is a continuously differentiable vector function $x(t)$ on $[0,T]$ which satisfies the system of the ordinary differential equations (1) on $[0,T]$ and the multipoint condition (2).

Let us now investigate boundary value problem (1), (2) by the parametrization method. Divide the interval $[0,T]$ into subintervals: $[0,T)=\bigcup\limits_{r=1}^{N}[t_{r-1},t_r)$.

Let $x_r(t)$ be the restriction of function $x(t)$ to the r-th interval $[t_{r-1},t_r)$, i.e. $x_r(t)=x(t)$, for $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$. Then problem (1), (2) is reduced to the equivalent multipoint boundary value problem

$$\frac{dx_r}{dt}=A(t)x_r+f(t),\ t\in[t_{r-1},t_r),\ \ r=\overline{1,N},\qquad (3)$$

$$\sum_{j=0}^{N-1}B_jx_{j+1}(t_j)+B_N\lim_{t\to T-0}x_N(t)=d,\quad d\in R^n,\qquad (4)$$

$$\lim_{t\to t_s-0}x_s(t)=x_{s+1}(t_s),\ s=\overline{1,N-1},\ (5)$$

where (5) are conditions for matching the solution at the interior points of partition $[0,T]$.

$C([0,T],N,R^{nN})$ is the space of function systems $x[t]=(x_1(t),x_2(t),...,x_N(t))$, where $x_r:[t_{r-1},t_r]\to R^n$ arecontinuous and have finite left-sided limits $\lim\limits_{t\to t_r-0}x_r(t)$, for all $r=\overline{1,N}$, with the norm $\|x[\cdot]\|_2=\max\limits_{r=\overline{1,N}}\sup\limits_{t\in[t_{r-1},t_r)}\|x_r(t)\|$.

A solution of problem (3)-(5) is a system of function $x[t]=(x_1(t),x_2(t),...,x_N(t))\in C([0,T],N,R^{nN})$ with continuously differentiable functions $x_r(t)$, $r=\overline{1,N}$, on $[t_{r-1},t_r)$ satisfying, the system of ordinary differential equations (3) and conditions (4), (5).

Introducing the additional parameters $\lambda_r=x_r(t_{r-1})$, $r=\overline{1,N}$, $\lambda_{N+1}=x_N(t_N)$, and performing a replacement of the function $u_r(t)=x_r(t)-\lambda_r$ on each r-th interval, we obtain the boundary value problem with parameters:

$$\frac{du_r}{dt}=A(t)[u_r(t)+\lambda_r]+f(t),\ t\in[t_{r-1},t_r),\ \ r=\overline{1,N},\ (6)$$

$$u_r(t_{r-1})=0,\quad r=\overline{1,N},\ (7)$$

$$\sum_{j=0}^{N}B_j\lambda_{j+1}=d,\quad d\in R^n,\ (8)$$

$$\lambda_s+\lim_{t\to t_s-0}u_s(t)=\lambda_{s+1},\ s=\overline{1,N}.\ (9)$$

A solution of problem (6) - (9) is a pair $(\lambda,u[t])$ with elements $\lambda=(\lambda_1,\lambda_2,...,\lambda_{N+1})\in R^{n(N+1)}$, $u[t]=(u_1(t),u_2(t),...,u_N(t))\in C([0,T],N,R^{nN})$ arecontinuously differentiable on $[t_{r-1},t_r)$, $r=\overline{1,N}$, and satisfy the system of the ordinary differential equations (6) and the conditions (7) - (9) at the $\lambda_r=\lambda_r^*$, $r=\overline{1,N+1}$.

The number of entered parameters will be one more than the numberof unknown functions. This is due to the fact that in the givenscheme of the parametrization method one more parameter is addedat the point $t=T$.

Problems (1), (2) and (6) - (9) are equivalent. If $x(t)$ is a solution to problem (1), (2), then the pair $(\lambda, u[t])$ where $\lambda=(x_1(t_0), x_2(t_1),..., x_N(t_N))$, $u[t]=(x_1(t)-x_1(t_0), x_2(t)-x_2(t_1),..., x_N(t)-x_N(t_{N-1}))$, is a solution to problem (6)-(9). Conversely, if the pair $(\tilde{\lambda}, \tilde{u}[t])$ with elements $\tilde{\lambda}=(\tilde{\lambda}_1, \tilde{\lambda}_2,...,\tilde{\lambda}_{N+1})$, $\tilde{u}[t]=(\tilde{u}_1(t), \tilde{u}_2(t),..., \tilde{u}_N(t))$, is a solution to problem (6)-(9), then the function $\tilde{x}(t)$ defined by the equalities $\tilde{x}(t)=\tilde{u}_r(t)+\tilde{\lambda}_r$, $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$, $\tilde{x}(T)=\tilde{\lambda}_{N+1}$, is a solution to the origin boundary value problem (1), (2).

Using the fundamental matrix $\Phi(t)$ of differential equation $\frac{dx}{dt}=A(t)x$ on $[t_{r-1},t_r)$, $r=\overline{1,N}$, we reduce the Cauchy problem for the system of ordinary differential equations with parameters (6), (7) to the equivalent system of integral equations:

$$u_r(t)=\Phi(t)\int_{t_{r-1}}^{t}\Phi^{-1}(\tau)[A(\tau)\lambda_r+f(\tau)]d\tau, \quad t\in[t_{r-1},t_r),\ r=\overline{1,N}. \quad (10)$$

Solving (10), we find a representation of $u_r(t)$ in terms of $\lambda\in R^{n(N+1)}$ and $f(t)$. Substituting theminto (8) and (9) yields a system of algebraic equations for finding the unknown parameters λ_r, $r=\overline{1,N+1}$:

$$\sum_{j=0}^{N}B_j\lambda_{j+1}=d, \quad (11)$$

$$\lambda_s+\Phi(t_s)\int_{t_{s-1}}^{t_s}\Phi^{-1}(\tau)A(\tau)\lambda_s d\tau-\lambda_{s+1}=-\Phi(t_s)\int_{t_{s-1}}^{t_s}\Phi^{-1}(\tau)f(\tau)d\tau,\ s=\overline{1,N}. \quad (12)$$

Denote the matrix correspondingto the left-hand side of the system of (11), (12) by Q, the system can be written as

$$Q\lambda=-F, \quad (13)$$

where

$$F=\left(-d,\ \Phi(t_1)\int_{0}^{t_1}\Phi^{-1}(\tau)f(\tau)d\tau,...,\Phi(t_{N-1})\int_{t_{N-2}}^{t_{N-1}}\Phi^{-1}(\tau)f(\tau)d\tau,\ \Phi(T)\int_{t_{N-1}}^{T}\Phi^{-1}(\tau)f(\tau)d\tau\right)'.$$

It is not difficult to establish that the solvability of the boundary value problem (1), (2) is equivalent to the solvability of the system (13).

The solution of system (13) the vector $\lambda^*=(\lambda_1^*, \lambda_2^*,..., \lambda_{N+1}^*)\in R^{n(N+1)}$ consists of the initial value problem (1), (2) in the initial points of sub-intervals, i.e., $\lambda_r^*=x_r^*(t_{r-1})$, $r=\overline{1,N}$, $\lambda_{N+1}^*=x_N^*(t_N)$.

If you know the $\lambda^*=(\lambda_1^*, \lambda_2^*,..., \lambda_{N+1}^*)$ is solution of the system (13), then the solution of the problem (1), (2) is determined by the equalities:

$$x^*(t)=\Phi(t)\Phi^{-1}(t_{r-1})\lambda_r^*+\Phi(t)\int_{t_{r-1}}^{t}\Phi^{-1}(\tau)f(\tau)d\tau, \quad t\in[t_{r-1},t_r), \quad r=\overline{1,N}, \quad (14)$$

$$x^*(T)=\Phi(T)\Phi^{-1}(t_{N-1})\lambda_N^*+\Phi(T)\int_{t_{N-1}}^{T}\Phi^{-1}(\tau)f(\tau)d\tau. \quad (15)$$

Thus, in this case, we obtain the solution of the linear multipoint boundary value problem for system of ordinarydifferential equations (1), (2) in an analytical form (14), (15).
We give an example to show that our assumptions can be satisfied.

Example. Consider on $[0,1]$ the linear three-point boundary value problem for system of ordinary differential equations:

$$\frac{dx}{dt} = A(t)x + f(t),\ t \in [0,1],\ x \in R^2, \qquad (16)$$

$$B_0 x(t_0) + B_1 x(t_1) + B_2 x(t_2) = d, \quad d \in R^2, \ (17)$$

where $A(t) = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$, $\quad f(t) = \begin{pmatrix} -t^3 + 2t - 2 \\ -t^3 + t^2 \end{pmatrix}$, $\quad B_0 = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$, $\quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\quad B_2 = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$,

$d = \begin{pmatrix} -\frac{27}{4} \\ -\frac{17}{8} \end{pmatrix}$, $t_0 = 0, t_1 = \frac{1}{2}, t_2 = 1$.

In this example a fundamental matrix of differential part is $\Phi(t) = \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix}$.

Let us now investigate boundary value problem (16), (17) by the parametrization method. We introduce the additional parameters $\lambda_1 \hat{=} x(0)$, $\lambda_2 \hat{=} x\left(\frac{1}{2}\right)$, $\lambda_3 \hat{=} x(1)$, and performing a replacement of the functions $u_1(s) = x(s) - \lambda_1$, $s \in \left[0, \frac{1}{2}\right)$, $u_2(s) = x(s) - \lambda_2$, $s \in \left[\frac{1}{2}, 1\right)$.Then for the function $u_r(s),\quad r = 1,2,$ we have the equalities

$$u_1(t) = \begin{pmatrix} -\frac{1}{3}\left(3 - e^{2t} - 2e^{-t}\right) & \frac{1}{3}\left(e^{2t} - e^{-t}\right) \\ \frac{2}{3}\left(e^{2t} - e^{-t}\right) & -\frac{1}{3}\left(3 - 2e^{2t} - e^{-t}\right) \end{pmatrix}\lambda_1 + \begin{pmatrix} t^2 - 1 - \frac{e^{2t}}{3} + \frac{4e^{-t}}{3} \\ t^3 + 2 - \frac{2e^{2t}}{3} - \frac{4e^{-t}}{3} \end{pmatrix}, \quad t \in \left[0, \frac{1}{2}\right),$$

$$u_2(t) = \begin{pmatrix} \frac{1}{3}\left(e^{2t-1} + 2e^{-t+\frac{1}{2}} - 3\right) & -\frac{1}{3}\left(e^{-t+\frac{1}{2}} - e^{2t-1}\right) \\ -\frac{2}{3}\left(e^{-t+\frac{1}{2}} - e^{2t-1}\right) & \frac{1}{3}\left(2e^{2t-1} + e^{-t+\frac{1}{2}} - 3\right) \end{pmatrix}\lambda_2 + \begin{pmatrix} t^2 - 1 - \frac{11e^{2t-1}}{24} + \frac{29e^{-t+\frac{1}{2}}}{24} \\ t^3 + 2 - \frac{22e^{2t-1}}{24} - \frac{29e^{-t+\frac{1}{2}}}{24} \end{pmatrix}, \quad t \in \left[\frac{1}{2}, 1\right].$$

Boundary condition and matchingconditions of solution at $t = 1/2$, $t = 1$lead to the following system oflinear algebraic equations with respect to parameters:

$$Q\lambda^* = F, \quad \lambda^* \in R^6, \ (18)$$

where

$$Q = \begin{pmatrix} 2 & 1 & 1 & 0 & 3 & -2 \\ 1 & -1 & 0 & -1 & 0 & 1 \\ \frac{1}{3}\left(e + 2e^{-\frac{1}{2}}\right) & \frac{1}{3}\left(e - e^{-\frac{1}{2}}\right) & -1 & 0 & 0 & 0 \\ \frac{2}{3}\left(e - e^{-\frac{1}{2}}\right) & \frac{1}{3}\left(2e + e^{-\frac{1}{2}}\right) & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{3}\left(e + 2e^{-\frac{1}{2}}\right) & \frac{1}{3}\left(e - e^{-\frac{1}{2}}\right) & -1 & 0 \\ 0 & 0 & \frac{2}{3}\left(e - e^{-\frac{1}{2}}\right) & \frac{1}{3}\left(2e + e^{-\frac{1}{2}}\right) & 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} -\frac{27}{4} \\ -\frac{17}{8} \\ \frac{1}{12}\left(9 + 4e - 16e^{-\frac{1}{2}}\right) \\ -\frac{1}{24}\left(51 - 16e - 32e^{-\frac{1}{2}}\right) \\ \frac{1}{24}\left(11e - 29e^{-\frac{1}{2}}\right) \\ \frac{1}{24}\left(22e - 72 + 29e^{-\frac{1}{2}}\right) \end{pmatrix}.$$

From (18) we find $\lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*) \in R^6$ with $\lambda_1^* = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\lambda_2^* = \begin{pmatrix} -\frac{3}{4} \\ \frac{17}{8} \end{pmatrix}$, $\lambda_3^* = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

In accordance with (14), (15) we find a unique solution to Problem (16), (17)

$$x^*(t) = \begin{pmatrix} \frac{1}{3}(e^{2t} - 4e^{-t}) \\ \frac{2}{3}(e^{2t} + 2e^{-t}) \end{pmatrix} + \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix} \int_0^t \begin{pmatrix} \frac{1}{3}(2\tau - 2\tau^3 + \tau^2 - 2)e^{-2\tau} \\ \frac{1}{3}(4\tau - \tau^3 - \tau^2 - 4)e^{\tau} \end{pmatrix} d\tau = \begin{pmatrix} t^2 - 1 \\ t^3 + 2 \end{pmatrix}, \quad t \in \left[0, \frac{1}{2}\right),$$

$$x^*(t) = \begin{pmatrix} \frac{1}{24}\left(11e^{2t-1} - 29e^{-t+\frac{1}{2}}\right) \\ \frac{1}{24}\left(22e^{2t-1} + 29e^{-t+\frac{1}{2}}\right) \end{pmatrix} +$$

$$+ \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix} \int_{\frac{1}{2}}^t \begin{pmatrix} \frac{1}{3}(2\tau - 2\tau^3 + \tau^2 - 2)e^{-2\tau} \\ \frac{1}{3}(4\tau - \tau^3 - \tau^2 - 4)e^{\tau} \end{pmatrix} d\tau = \begin{pmatrix} t^2 - 1 \\ t^3 + 2 \end{pmatrix}, \quad t \in \left[\frac{1}{2}, 1\right],$$

i.e. $x^*(t) = \begin{pmatrix} t^2 - 1 \\ t^3 + 2 \end{pmatrix}, \quad t \in [0,1].$

In this example, we have been able to construct the fundamental matrix of the differential part considered ordinary differential equation. Algorithm of the parameterization method allowed to construct a solution of problem (16), (17) in an explicit form.

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ЖӘЙ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІ ҮШІН СЫЗЫҚТЫ КӨПНҮКТЕЛІ ШЕТТІК ЕСЕПТІҢ ШЕШІЛІМДІЛІГІ ТУРАЛЫ

Ж.М. Қадырбаева[1]
[1]ф.-м.ғ.к., аға оқытушы,
Қазақ мемлекеттік қыздар педагогикалық университеті,
Қазақстан, Алматы қ., e-mail: apelman86pm@mail.ru

Жәй дифференциалдық теңдеулер жүйесі үшін сызықты көпнүктелі шеттік есеп қарастырылады. Қарастырылып отырған есепті шешу үшін параметрлеу әдісі қолданылады. Жәй дифференциалдық теңдеулер жүйесі үшін сызықты көпнүктелі шеттік есеп шеткі жәнеаралықты бөлунүктелерінде қосымша параметрлер енгізу арқылы параметрлі пара-пар шеттік есепке келтіріледі. Мақала жәй дифференциалдық теңдеулер жүйесі үшін сызықты үшнүктелі шеттік есептің шешімін табу мысалымен сипатталады.

Түйін сөздер: дифференциалдық теңдеу, шеттік есеп, параметрлеу әдісі, шешілімділік, фундаменталдық матрица.

О РАЗРЕШИМОСТИ ЛИНЕЙНОЙ МНОГОТОЧЕЧНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ СИСТЕМ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Ж.М. Кадирбаева[1]
[1] К.ф.-м.н., старший преподаватель,
Казахский государственный женский педагогический университет,
Казахстан, г.Алматы, e-mail: apelman86pm@mail.ru

Рассматривается линейная многоточечная краевая задача для систем обыкновенных дифференциальных уравнений. Для решения рассматриваемой задачи применяется метод параметризации. Линейная многоточечная краевая задача для систем обыкновенных дифференциальных уравнений путем введения дополнительных параметров на концах и в точках разбиения интервала сводится к эквивалентной краевой задаче с параметрами. Статья иллюстрируется примером для нахождения решения линейной трехточечной краевой задачи для систем обыкновенных дифференциальных уравнений.

Ключевые слова: дифференциальное уравнение, краевая задача, метод параметризации, разрешимость, фундаментальная матрица.